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Crossover temperature in electron–phonon heat exchange in layered nanostructures

Dragos-Victor Anghel\textsuperscript{1,5} \textsuperscript{\textcopyright}, Claudiu Caraiani\textsuperscript{2} and Yuri M Galperin\textsuperscript{3,4}

\textsuperscript{1} Horia Hulubei National Institute for R&D in Physics and Nuclear Engineering, Măgurele, Romania
\textsuperscript{2} University of Bucharest, Faculty of Physics, Măgurele, Romania
\textsuperscript{3} Department of Physics, University of Oslo, PO Box 1048 Blindern, 0316 Oslo, Norway
\textsuperscript{4} A. F. Ioffe Physico-Technical Institute of Russian Academy of Sciences, 194021 St. Petersburg, Russia

E-mail: dragos@theory.nipne.ro

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Abstract
We study theoretically the effect of the effective dimensionality of phonon gas distribution on heat exchange between electrons and phonons in layered nanostructures. If we denote the electron temperature by $T_e$ and the phonon temperature by $T_{ph}$, then the total heat power $P$ is proportional—in general—to $T_e^x - T_{ph}^x$, the exponent $x$ being dependent on the effective dimensionality of the phonon gas distribution. If we vary the temperature in a wide enough range, the effective dimensionality of the phonon gas distribution changes going through a crossover around some temperature, $T_C$. These changes are reflected by a change in $x$. On the one hand, in a temperature range well below the crossover temperature $T_C$, only the lowest branches of the phonon modes are excited. They form a (quasi-)two-dimensional gas, with $x = 3.5$. On the other hand, well above $T_C$, the phonon gas distribution is quasi-three-dimensional and one would expect to recover three-dimensional results, with $x = 5$. But this is not the case in our layered structure. The exponent $x$ has a complicated, non-monotonic dependence on temperature forming a ‘plateau region’ just after the crossover temperature range, with $x$ between 4.5 and 5. After the plateau region, $x$ decreases, reaching values between 3.5 and 4 at the highest temperature used in our numerical calculations, which is almost two orders of magnitude higher than $T_C$.

Keywords: nanosystems, thermoelectric systems, thermoelectric effects, mesoscopics

(Some figures may appear in colour only in the online journal)

1. Introduction

Continuous efforts in miniaturizing devices require better understanding of their quantum behaviors at nanoscale lengths and sub-kelvin temperatures. Such devices include microrefrigerators, microbolometers, and microcalorimeters [1]. Microrefrigerators have been proven to be able to reduce the temperature of the electron gas of a normal metal island down to 30 mK, starting from a bath temperature of 150 mK [2]. Such an experimental achievement allows the use of these devices in refrigerating microbolometers, microcalorimeters, qubits, etc.

The devices are usually layered structures consisting of thin metal films (for example, Cu, Al, etc) deposited on free-standing dielectric membranes, typically made of silicon nitride (SiN).

The membrane is itself supported by a bulk substrate, which may be considered the thermostat for the whole device. A central issue in the understanding of how the devices function and how they respond to the absorption of electromagnetic radiation is the interaction and heat exchange between the electron system and the phonon system in normal metal films. Our study is based on the model presented in [3, 4], in which a Cu layer of thickness $d$ is deposited on a SiN membrane of thickness $L - d$, such that the total thickness of the system is $L$—as one can see in figure 1.

At temperatures of the order of 1 K, due to the depopulation of phonon modes and the degeneracy of the electron system, the two subsystems of a normal metal—the electrons and the phonons—become almost isolated from each other (see, for example, the discussion in section 2 of [1]). For this reason, they can ‘equilibrate’ independently, that is, they may reach equilibrium distributions over the quasiparticle states at some effective
temperatures that we shall denote by $T_e$ (for electrons) and $T_{ph}$ (for phonons). Note that we do not aim at the calculation of the temperatures $T_e$ and $T_{ph}$, which may differ from the temperature of the bulk substrate (the thermostat). Therefore we do not need to include phonon–electron scattering explicitly. Using the effective temperatures, the heat exchange $P$ between the electrons and the phonons may be written in most cases as [5–8]

$$P = V \Sigma(T_e^x - T_{ph}^x),$$

where $V$ is the s-dimensional ‘volume’ and $\Sigma$ is a constant of the material. The exponent $x$ varies, depending on the model and the dimensionality of the electrons and the phonon gases. For 3D models, $x = 5$ [5]; for quasi-2D ones, $x = 4$ [6]; and for a 1D phonon gas coupled to a 3D electron system, $x = 3$ [8]. Apparently, this leads to the conjecture

$$x = 2 + s$$

where $s$ is the dimensionality of the phonon gas.

Apart from equation (1b), it was observed that $s$ may also take fractional values [3, 4]. In mesoscopic slabs, as in the devices described here, the phonons do not represent pure 2D or 3D quantized vibrations, but superpositions of waves with different polarizations. Similarly, in electrons dispersion may represent a superposition of lower-dimensional conduction bands (in slabs, it is a superposition of 2D bands), which may not share the same volume as the phonon system, as we see in figure 1. In such cases, the conjecture (1b) is not applicable and the results are qualitatively different [3, 4], as we shall also see below.

The temperature range in which these devices operate is several hundreds of millikelvins [1, 2]. For typical values of $L$ (of the order of 100 nm) and for the sound velocities of SiN (the longitudinal and transversal sound velocities are $c_l \approx 10\,300\,\text{m}\,\text{s}^{-1}$ and $c_t \approx 6500\,\text{m}\,\text{s}^{-1}$, respectively), the dominant phonon wavelength of a 3D phonon gas model is comparable to $L$ at temperatures of the order of 100 mK. As the temperature decreases, the 3D dominant phonon wavelength becomes much bigger than $L$ and excitations of phonon modes perpendicular to the membrane surfaces are suppressed. In such a case, we can say that the phonon gas is 2D. At the other extreme, at high enough temperatures, the dominant phonon wavelength may become much smaller than $L$ and the phonon gas is practically 3D [9–12]. The crossover temperature may be estimated as $T_C = c_l h / (2k_B L) \approx 237\,\text{mK}$.

To make the argument more quantitative, a typical quantity that provides information about the dimensionality of the phonon gas distribution is the exponent of the temperature dependence of the heat capacity $C_V \equiv \partial U / \partial T$ (where $U$ is the internal energy of the phonon gas) [11, 12]. The phonon modes in slabs are grouped into three polarizations: horizontal shear ($b$), symmetric Lamb modes ($s$) and antisymmetric Lamb modes ($a$) (see [13] for more details). The phonons do not interact with one another (that is, we do not consider nonlinear effects), so the internal energy may be written as a sum $U = U_b + U_s + U_a$, where $U_b$, $U_s$, and $U_a$ represent the independent contributions from the $b$, $s$, and $a$ modes. Therefore, the heat capacity may be written as $C_V \equiv \partial(U_b + U_s + U_a) / \partial T \equiv C_{Vb} + C_{Vs} + C_{Va}$. Denoting, in general, the polarization as $\alpha = h, s, a$, we obtain

$$C_V = \frac{\partial}{\partial T} \left[ \sum_{\alpha} \int_0^{\infty} \frac{\sigma_\alpha(\epsilon) \epsilon \, d\epsilon}{e^{\epsilon/k_B T} - 1} \right],$$

where $p_\alpha$ represents the phonon modes of a specific polarization $\alpha$. If we can transform the summation over $p_\alpha$ into an integral over the phonon energies $\epsilon$, by introducing a density of states (DOS) $\sigma_\alpha(\epsilon)$ for each polarization (eventually as asymptotic 2D and 3D limits, as we shall see below), then $C_V$ of equation (2) may be approximated by

$$C_V = \frac{\partial}{\partial T} \left[ \int_0^{\infty} \frac{\sigma_\alpha(\epsilon) \epsilon \, d\epsilon}{e^{\epsilon/k_B T} - 1} \right],$$

If, furthermore,

$$\sigma_\alpha(\epsilon) = C_\alpha \epsilon^{\nu_\alpha},$$

then we can rewrite (2) as

$$C_V = \sum_{\alpha} \left\{ C_\alpha \frac{\partial}{\partial T} \left[ (k_B T)^{\nu_{\alpha} + 2} \int_0^{\infty} y^{\nu_{\alpha} + 1} \, dy \right] \right\}$$

$$= \sum_{\alpha} \left\{ k_B C_\alpha \Gamma(r + 3) \zeta(r + 2) (k_B T)^{\nu_{\alpha} + 1} \right\},$$

where $y \equiv \epsilon / (k_B T)$ and $\zeta$ is the Riemann zeta function. We can define the exponents of the temperature dependence,

$$x_C \equiv \frac{\partial \ln(C_V)}{\partial \ln(T)} \quad \text{and} \quad x_{C\alpha} \equiv \frac{\partial \ln(C_{V\alpha})}{\partial \ln(T)},$$

such that, if (4a) holds, we obtain

$$x_{C\alpha} = x_{\alpha} + 1$$

and, if $r_b = r_s = r_a \equiv r$,

$$x_C = r + 1.$$

In figure 2 we plot the exponents $x_C$ and $x_{C\alpha}$, as defined by equation (5), in both linear (a) and logarithmic (b) scales, to emphasize the $T_{ph} > T_C$ (a) and $T_{ph} < T_C$ (b) ranges, respectively (notice that the temperature range of interest for us is well below the Debye temperature, which is of the order of 1000 K [14, 15]). We observe that all exponents converge to 3 as $T_{ph}$ increases [9–12]. According to equations (6) and (7), this leads to the conclusion that for each polarization, the DOS converges asymptotically to $\sigma \sim \epsilon^2$, at high energies, approaching the 3D DOS. At low
$T$, $x_{C_h}$ and $x_{C_s}$ converge to 2, whereas $x_{C_a}$ converges to 1. This happens because of the dispersion relations of the phonon modes. At long wavelengths (much longer than $L$), for the $h$ and $s$ modes $\epsilon \propto k$, which leads to a 2D DOS $\sigma_{h,s} \propto \epsilon$, whereas for the $a$ modes $\epsilon \propto k^2$, which leads to a 2D DOS $\sigma = \text{const.}$, as in the case of massive particles [11, 12]. We also observe in figure 2 that crossover from the low-$T$ 2D phonon distribution to the high-$T$ 3D phonon distribution takes place in a temperature range around $T_C$, as qualitatively explained above.

The exponent of the temperature dependence of the heat capacity provides us with a direct tool to evaluate the phonon DOS and, through it, the dimensionality of the phonon gas distribution. Besides this, the phonon gas distribution may be probed by the exponent of the temperature dependence of the heat conductivity, as was done in [9–12], only in this case, the exponent is determined, besides the phonon gas dimensionality, by the scattering mechanisms of the phonons.

Similar to $x_C$ and $x_{C_a}$, the exponent $x$ of equation (1) depends on the effective dimensionality of the phonon gas. Therefore, in the temperature range where the dimensionality of the phonon gas changes, the exponent $x$ may also change, showing a crossover between values corresponding to different dimensionalities. This would allow us to make a link between models for different dimensionalities, verify consistently the connection between such models, and find their ranges of validity. These issues, which, to the best of our knowledge, have not been investigated, will be explored here, as a continuation of previous studies [3, 4].

The paper is organized as follows. In the next section we present the theory, with the system’s specifications; then, we present the results; and finally, we draw conclusions. In the following, we try to preserve the notations from [3, 4] as much as possible.

2. Methods
As in [3, 4], we consider a metallic film (for example, Cu) deposited on a free-standing dielectric membrane (for example silicon nitride) and perfectly glued to it [16, 17], as shown in figure 1. For the mathematical description, we choose a Cartesian system of coordinates, such that all the surfaces of the membrane and of the film are parallel to the $(xy)$ plane. In the $z$ direction, the membrane extends in the interval $[-L/2, L/2 - d]$, whereas the metal occupies the interval $[L/2 - d, L/2]$. In the $x$ and $y$ directions, the system is considered to be very large (in comparison with the wavelengths of the quasiparticles involved) and has area $A$; we assume that $L \ll \sqrt{A}$.  

2.1. Description of the electrons
The metal contains electrons, which interact with the phonons that propagate in the whole system. The electrons’ wave functions propagate in the $(xy)$ plane and along the $z$ direction. We denote the electrons’ wavevector as $\mathbf{k} \equiv (k_x, k_y)$, where $k_x$ and $k_y$ are the components of $\mathbf{k}$ perpendicular and parallel to the $z$ axis, respectively. For $k_{2d}$, we impose periodic boundary conditions in the $(xy)$ plane, whereas for $k_z$ we impose Dirichlet boundary conditions at $z = L/2 - d$ and $z = L/2$. This leads to a density of (allowed quantum) states in the $k_{2d}$ variable equal to $A/(2\pi)^2$ and to the quantization condition $k_z \equiv n\pi/d$, where $n$ is an integer. Then, the electrons’ wavefunction is of the form

$$\psi_{k_{2d},k_z}(\mathbf{r}, t) \equiv \psi_{k_{2d}}(\mathbf{r}, t) = \phi_{k_z}(z)e^{i\mathbf{k}_{2d} \cdot \mathbf{r}/\hbar}/\sqrt{A},$$

where $\phi_{k_z}(z) = \frac{2}{d} \sin \left(\frac{2}{d} \left( z + \frac{L}{2} \right) k_z \right)$ and the electrons’ energy is

$$\varepsilon_k = \frac{\hbar^2 k^2}{2m_e} \equiv \frac{\hbar^2 k_x^2}{2m_e} + \frac{\hbar^2 k_y^2}{2m_e} \equiv \varepsilon_{k_{2d}},$$
where \( m_e \) is the electron effective mass. In these notations, the electron annihilation and creation field operators are

\[
\Psi^\dagger(\mathbf{r}, t) = \sum_{\mathbf{k}, \mathbf{k}'} \psi_{\mathbf{k}, \mathbf{k}'}(\mathbf{r}, t) c_{\mathbf{k}, \mathbf{k}'}
\]

and

\[
\Psi(\mathbf{r}, t) = \sum_{\mathbf{k}, \mathbf{k}'} \psi_{\mathbf{k}, \mathbf{k}'}^*(\mathbf{r}, t) c_{\mathbf{k}', \mathbf{k}}^\dagger,
\]

respectively, where \( c_{\mathbf{k}, \mathbf{k}'} \) and \( c_{\mathbf{k}', \mathbf{k}}^\dagger \) are the electron annihilation and creation operators on the state \( \psi_{\mathbf{k}, \mathbf{k}'} \).

Since \( \sqrt{\lambda} \gg L > d \), \( \mathbf{k} \equiv (\mathbf{k}_x, \mathbf{k}_y) \) has a quasi-continuous spectrum, whereas \( \mathbf{k}_z \) will be considered discrete. States of constant \( k_z \) (or \( n \)) and any value of \( k \) form the 2D conduction bands. We denote by \( n_p \) the number of occupied 2D bands at \( T_c = 0 \). K.

\[
n_p \equiv \left\lfloor \frac{2m_e E_F}{\pi \hbar^2} d \right\rfloor.
\]

where \( \lfloor \cdot \rfloor \) is the biggest integer smaller than or equal to \( x \) and \( E_F \) is the Fermi energy. If the normal metal is Cu, we use the values \( E_F = 7 \text{ eV} \) and \( m_e = m_{e0}, \) where \( m_{e0} \) is the free electron mass [18] (although on page 48 of [18] \( m_e = 1.3m_{e0}, \) we choose \( m_e = m_{e0} \) for clarity and because this choice does not lead to any qualitative differences). Then, for \( d = 10 \text{ nm} \), we get \( n_p = 43 \), whereas for \( d = 20 \text{ nm} \), we get \( n_p = 86 \).

### 2.2. Description of the phonons

The phonons are the quanta of elastic vibrations of the system. They propagate in the \((xy)\) plane and are stationary waves along \( z \). At the surfaces of the system, at \( z = \pm L/2 \), the phonons satisfy free boundary conditions. To simplify the calculation of the phonon modes [3, 4], we assume that the metal and the dielectric membrane have identical mechanical properties, namely they have the same mass density and identical elastic properties. Although this seems to be a crude simplification, it gives us a method to study the characteristics of electron-phonon interaction in general.

The phonon modes are eigenstates of the elastic dynamic equations and are grouped into three polarizations: horizontal shear (\( h \)), symmetric (\( s \)) and antisymmetric (\( a \)) modes [13]. While the \( h \) modes are purely transversal (t) vibrations, the \( s \) and \( a \) modes are superpositions of transversal and longitudinal (l) vibrations [13]—in a longitudinal vibration, the displacement field is in the propagation direction, whereas in a transversal vibration the displacement field and the propagation direction are perpendicular to each other.

An \( h \) mode is characterized by a single wavevector, of components \( q_h, q_h \), perpendicular and parallel to the \( z \) axis, respectively. The components of \( q_h \) satisfy periodic boundary conditions in the \((xy)\) plane and \( q_{h} \equiv \nu \pi / L \), where \( \nu = 0, 1, \ldots \). States of constant \( \nu \) and variable \( q_h \) define 2D phonon branches.

An \( s \) mode—and, similarly, an \( a \) mode—is characterized by two wavevectors, corresponding to the \( l \) and \( t \) vibrations. The wavevector corresponding to the longitudinal vibration has the components \( q_l \) and \( \bar{q}_l \), whereas the wavevector of the transversal vibration has the components \( q_t \) and \( \bar{q}_t \). The component \( q_t \), perpendicular to the \( z \) axis, is the same for both \( l \) and \( t \) vibrations, whereas the components \( \bar{q}_l \) and \( \bar{q}_t \), parallel to the \( z \) axis, are, in general, different. As in the case of electrons, \( q_l \) satisfies periodic boundary conditions on the area \( A \) and has a DOS of \( A/(2\pi)^2 \). The free boundary conditions imposed at \( z = \pm L/2 \) lead to the equation [4, 13]

\[
-4q_t^2 \bar{q}_l \bar{q}_t = \tan (\bar{q}_t L/2) \quad \tan (\bar{q}_t L/2) = \tan (\bar{q}_t L/2)^{-1},
\]

for the components \( \bar{q}_l \) and \( \bar{q}_t \), where \( q_l \equiv |q_t| \). The exponents \( +1 \) and \( -1 \) on the right-hand side of equation (11a) correspond to the polarizations \( s \) and \( a \), respectively [13], whereas \( \xi \equiv (\alpha, \nu) \) is a doublet, containing the polarization \( \alpha \) (which, in equation (11a), is only \( s \) or \( a \)) and the branch number \( \nu = 0, 1, 2, \ldots \) (as in the case of the \( h \) polarization), as we shall specify below. If we denote by \( c_l \) and \( c_t \) the longitudinal sound velocity and the transversal sound velocity, respectively, then the angular frequency, common to both longitudinal and transversal vibrations, is given by

\[
\omega_{l,t} = c_l \sqrt{\bar{q}_l^2 + q_t^2} = c_t \sqrt{\bar{q}_l^2 + q_t^2}.
\]

The system (11) has a countable, infinite number of solutions, for each \( \alpha \) and \( q_l \). These solutions are denoted by \( r \) and form the branches when plotted as functions of \( q_l \). In general, \( \bar{q}_l \) and \( \bar{q}_t \) may take both (positive) real and imaginary values, as explained for example in [13] (see also [19]), whereas \( q_l \) takes only real positive values, by definition. Therefore, we introduce the notations \( \bar{q}_{l,s} \equiv \bar{q}_l + iq_l \), \( \bar{q}_{l,a} \equiv q_l + iq_l \), in which either \( q_l \) or \( p \) may be different from zero, but not both at the same time or in the same polarization (see [19] for details). If \( p \) is the mass density of the system (considered homogeneous in the whole volume), then the sound velocities may be expressed in terms of the Lamé coefficients \( \lambda \) and \( \mu \) [13],

\[
c_l^2 = \frac{\mu}{\rho}, \quad c_t^2 = \frac{\lambda + 2\mu}{\rho};
\]

we define \( J \equiv c_l^2 / c_t^2 \), which shall be used below.

We define the elastic modes by \( \mathbf{w}_{\mathbf{q}_l}(z)e^{i(\mathbf{q}_l \cdot \mathbf{r}) - \omega_{l,t} (t) / \sqrt{\lambda}} \), where the functions \( \mathbf{w}_{\mathbf{q}_l}(z) \) are normalized such that

\[
\int_{-L/2}^{L/2} \mathbf{w}_{\mathbf{q}_l}(z) \mathbf{w}_{\mathbf{q}_l}(z) \, dz = \delta_{\mathbf{q}_l, \mathbf{q}_l}.
\]

If we choose \( q_l \) along the \( z \) axis, then \( \mathbf{w}_{\mathbf{q}_l}(z) \) is in the \((x,z)\) plane, with the components [3, 4, 19]

\[
w_{q_l,q_l,r,t} = N_l \delta_{\mathbf{q}_l} \left[ 2q_t^2 \cos\left(\frac{\bar{q}_l L}{2}\right) \cos(\bar{q}_l z) \right.
\]

\[
+ \left( q_t^2 - q_l^2 \right) \cos\left(\frac{\bar{q}_l L}{2}\right) \cos(\bar{q}_l z) \right],
\]

\[
w_{q_l,q_l,r,t} = N_l q_l \left[ -2q_t \bar{q}_l \cos\left(\frac{\bar{q}_l L}{2}\right) \sin(\bar{q}_l z) \right.
\]

\[
+ \left( q_t^2 - q_l^2 \right) \cos\left(\frac{\bar{q}_l L}{2}\right) \sin(\bar{q}_l z) \right].
\]
To simplify the notations, in the following we shall write \( q_{\xi \ell} \) instead of \( q_{\xi, \ell} \) and \( q_t \) instead of \( q_{t, \xi} \), when this does not lead to confusion. Then, the normalization constants are

\[
\begin{align*}
\frac{1}{N_\ell} &= A \left[ 4|q_{\xi \ell}^2 - q_t^2| \cos \left( \frac{q_{\xi \ell} L}{2} \right) \sin \left( \frac{q_{\xi \ell} 
abla z}{2} \right) \right] \sin \left( \frac{q_{\xi \ell} 
abla z}{2} \right) \\
&= \frac{1}{N_\ell} \left[ 4|q_{\xi \ell}^2 - q_t^2| \cos \left( \frac{q_{\xi \ell} L}{2} \right) \sin \left( \frac{q_{\xi \ell} 
abla z}{2} \right) \right] \sin \left( \frac{q_{\xi \ell} 
abla z}{2} \right)
\end{align*}
\]

where \( a^\dagger_{q_{\xi \ell}} \) and \( a_{q_{\xi \ell}} \) are the phonon creation and annihilation operators.

The electron–phonon interaction is calculated in the deformation potential model, employing the interaction Hamiltonian [20],

\[
H_{\text{def}} = \frac{2}{3} E_F \int_{q_{\text{u}}=A \times d} d^4 r \, \nabla \cdot \mathbf{u}(r).
\]

Starting from equation (16) and applying the Fermi golden rule \( \Gamma_{ij} = (2\pi/\hbar)|f_i| |\langle H_{\text{def}} \rangle| f_j \delta(E_j - E_i) \) to calculate the transition rates \( \Gamma \) between the initial state \( |i\rangle \) of energy \( E_i \) and the final state \( |f\rangle \), of energy \( E_f \), the heat power was calculated in [3, 4] to be

\[
P \equiv P^{(0)}(T_e) - P^{(1)}(T_e, T_{ph}),
\]

where

\[
P^{(0)}(T_e) \equiv \frac{4\pi}{h} \sum_{k,k',n,n'} \frac{|\omega_{k,n}|}{\omega_{k',n'}} \frac{\gamma_{n,n'}^r}{\pi} \left[ f(\beta_e \epsilon_{k',n}) - f(\beta_e \epsilon_{k,n}) \right],
\]

\[
P^{(1)}(T_e, T_{ph}) \equiv \frac{4\pi}{h} \sum_{k,k',n,n'} \frac{|\omega_{k,n}|}{\omega_{k',n'}} \frac{\gamma_{n,n'}^r}{\pi} \left[ f(\beta_e \epsilon_{k',n}) - f(\beta_e \epsilon_{k,n}) \right],
\]

\[
P^{(2)}(T_e, T_{ph}) \equiv \frac{4\pi}{h} \sum_{k,k',n,n'} \frac{|\omega_{k,n}|}{\omega_{k',n'}} \frac{\gamma_{n,n'}^r}{\pi} \left[ f(\beta_e \epsilon_{k',n}) - f(\beta_e \epsilon_{k,n}) \right],
\]

where the normalization constants \( N_{\text{w}_{q_{\xi \ell}}} \) are given in equation (14). Explicitly, we obtain

\[
P^{(0)} = \frac{A}{\pi^2 \rho} \left( \frac{2}{3} E_F \right)^2 \left( \frac{d}{2} \right)^3 \times \sum_{\nu} \int_0^\infty dx \frac{d^4 x}{(2\pi)^4} n(\beta_e \hbar \omega_{q_{\xi \ell}}) \times \int_0^{2\pi} d\phi \int_0^\infty dy \gamma_{n,n'} [f(\beta_e \epsilon_{k',n'}) - f(\beta_e \epsilon_{k,n})] \times \delta(\epsilon_{k'-n'} - \epsilon_{k,n} + \hbar \omega_{q_{\xi \ell}}),
\]

If \( q \) has an arbitrary direction in the (xy) plane, then the components (13) have to be rotated accordingly.

With these definitions, the displacement field operator is

\[
\mathbf{u}(r, t) = \sum_{q_{\xi \ell}} \frac{\hbar}{2 \rho \omega_{q_{\xi \ell}}} e^{i(q_{\xi \ell} \cdot r - i\omega_{q_{\xi \ell}} t)} [a_{q_{\xi \ell}} w_{q_{\xi \ell}}(z) + a_{q_{\xi \ell}}^\dagger w_{q_{\xi \ell}}^*(z)],
\]

where

\[
\begin{align*}
H_{\text{def}} &= \frac{2}{3} E_F \int_{q_{\text{u}}=A \times d} d^4 r \, \nabla \cdot \mathbf{u}(r).
\end{align*}
\]
\[ P_{\alpha}^{(0)} = \frac{A}{\pi} \frac{1}{|n'|^3} \left( \frac{2}{E_0} \right)^{\frac{3}{2}} \left( \frac{d}{2} \right)^{-3} \]
\[ \times \sum_{\nu} \int_0^\infty dx_1 x_1 n(\beta_n \omega_{\nu n}) L_{\nu n}^{(0)}(x_1) \]
\[ \times \int_0^{2\pi} dx_0 \int_0^{\infty} dy'_0 |f[\beta_n(\epsilon_{k_1-q,n}')] - h_\omega_{\nu n}] - f(\beta_n \epsilon_{k_1,n})| \]
\[ \times \delta(\epsilon_{k_1-q,n}') - \epsilon_{k_1,n} + h_\omega_{\nu n}) \].

(19b)

where

\[ L_{\nu n}^{(0)}(x_1) = \sum_{n} \sum_{n'} |x_{1}^2 x_{1}^2 \cos(\pi n) |^2 G_{\alpha n,n'}(n, n')^2 h_{\omega, n,n'} \]
\[ \times \left\{ 4 |x_{1}^2 x_{1}^2 \cos(\pi n)|^2 \left[ (|x_{1}^2| + x_{1}^2) \sinh(2\chi_n) \right] \right. \]
\[ + (x_{1}^2 - |x_{1}^2|) \frac{\sin(2\chi_n)}{2\chi_n} \right\} \]
\[ + |x_{1}^2 - x_{1}^2| \cos(\pi n)]^2 \left[ (|x_{1}^2| + x_{1}^2) \sinh(2\chi_n) \right] \]
\[ - (|x_{1}^2| - x_{1}^2) \frac{\sin(2\chi_n)}{2\chi_n} \right\} \]
\[ - 4x_{1}^2 |\cos(\pi n)|^2 \left[ (|x_{1}^2| + x_{1}^2) \sinh(2\chi_n) \right] \]
\[ + x_{1}(|x_{1}^2| - x_{1}^2) \sin(2\chi_n) \]^{-1}.

(19c)

\[ G_{\alpha n,n'}(n, n') = \frac{2}{L} \int_{-L/2}^{L/2} dz \sin \left[ z - \frac{L}{2} + d \right] \frac{n \pi}{d} \]
\[ \times \sin \left[ z - \frac{L}{2} + d \right] \frac{n \pi}{d} \cos \left[ q_{\alpha n,n'}(z) \right] \]

(19d)

where

\[ \gamma_{\min} \equiv \frac{L}{4q_1} \frac{2m}{\hbar^2} \omega_{\nu n} + q_1^2 + [(n')^2 - n^2] \left( \frac{\pi}{L} \right)^2. \]
We observe that the integrals over $y_\parallel$ in equation (20) are identical:

\[
I_p \equiv \int_{\gamma_{\text{min}}}^{\infty} \frac{dy_\parallel y_\parallel}{\sqrt{y_\parallel^2 - y_\parallel^2_{\text{min}}}} \left\{ \frac{1}{e^{\frac{\mu}{kT} \left( \gamma_\parallel^2 + \left( \frac{\pi}{L} \right)^2 \right)} - e^{-\mu}} + 1 \right\}
\]

\[= \frac{1}{2} \sqrt{\frac{k_B T}{\hbar^2}} \frac{2m}{L} \int_{z_{\text{min}}}^{\infty} \frac{dz_\parallel}{\sqrt{z_\parallel^2 - z_\parallel^2_{\text{min}}}} \left\{ \frac{1}{e^{\frac{z_\parallel}{k_B T} - \gamma_\parallel^2 - z_\parallel^2_{\text{min}}} + 1} + \frac{1}{e^{\frac{z_\parallel}{k_B T} - \gamma_\parallel^2 - z_\parallel^2_{\text{min}}} + 1} \right\}
\]

(22)

where $z_\parallel \equiv \beta_k (h^2/2m)(2/L)\gamma_\parallel^2$, $z_\parallel_{\text{min}} \equiv \beta_k (h^2/2m)(2/L)\gamma_\parallel^2_{\text{min}}$, $z_{\text{ph}} \equiv \beta_k \hbar \omega_{l, k, q}$, and $z_\mu \equiv \beta_k \mu$.

If $z_{\text{ph}} - z_{\text{min}} \gg 1$ (as we shall see further, this condition is generally satisfied outside the crest region of figure 3), as was explained also in [3, 4], we can use the approximation (see the appendix of [4])

\[I_p \approx \frac{1}{2} \sqrt{\frac{k_B T}{\hbar^2}} \frac{2m}{L^2} \frac{z_{\text{ph}}}{\sqrt{z_{\text{ph}}^2 - z_{\text{min}}}} \frac{1}{e^{\frac{z_{\text{ph}}}{k_B T} - \gamma_\parallel^2 - z_{\text{min}}}}
\]

(23)

and observe that $I_p$ does not depend on $T_c$.

Since in the expressions for $P_{s}^{(1)}$ and $P_{a}^{(1)}$, $n(\beta_k \hbar c_0)$ replaces $n(\beta_k \hbar c)$, the fact that $I_p$ is independent of temperature implies that, in general, $P_{s}^{(1)}$ and $P_{a}^{(1)}$ are functions of $T_{ph}$ and are independent of $T_c$—this is also checked numerically and is found to be correct. Then, the exponents of the temperature dependence, as defined in (1a), are calculated as

\[x_{T_{ph}} = \frac{\partial \ln(P^{(1)})}{\partial \ln(T_{ph})} \quad \text{and} \quad x_{T_{e}} = \frac{\partial \ln(P^{(0)})}{\partial \ln(T_{e})}
\]

(24)

It can now be easily checked that equation (24) implies that $x_{T_{ph}} = x_{T_{e}} \equiv x$, which leads to the heat balance condition $P = 0$ for any $T_c = T_{ph}$.

3. Results

For concreteness, we consider systems of a total thickness $L = 100$ nm, of which the thickness of the metallic film is around 10 nm. The elastic properties of the entire system are due to the silicon nitride, which is a common material for the construction of the supporting membrane of nano-detectors [1]. For this material, the density is $\rho = 3290$ kg m$^{-3}$, whereas the longitudinal and transversal sound velocities are $c_1 = 10300$ m s$^{-1}$ and $c_1 = 6200$ m s$^{-1}$, respectively. As explained in the introduction, the phonon gas changes its effective dimensionality in a temperature range around $T_{C} \approx 237$ mK [11]. In the low-temperature limit ($T < T_{C}$) only the lowest $s$ and $a$ phonon branches contribute to thermodynamics and to the electron–phonon interaction, since the upper branches are depopulated [3, 4]. As the temperature increases, the phonon gas changes dimensionality, as the upper $s$ and $a$ branches gradually start to be populated and play a role in the interaction.

In the electron system, on the other hand, all bands, from 1 to $n_p$, are populated at any temperature and contribute—with the electrons close to the Fermi energy—to the electron–phonon interaction. To estimate the difference between the energies of electrons of the same $k_\parallel$, but belonging to different 2D conduction bands, for a metallic layer of thickness $d = 10$ nm, we calculate $(\epsilon_{k_\parallel=0, n=2} - \epsilon_{k_\parallel=0, n=1})/k_B \approx 131$ K and $(\epsilon_{k_\parallel=0, n=44} - \epsilon_{k_\parallel=0, n=43})/k_B \approx 3796$ K. We observe that this
difference is large compared to the energy of thermal phonons in a temperature range below 20 K, so electron scattering processes between different conduction bands are very rare and they do not significantly contribute to electron–phonon heat exchange (see also [3, 4]).

In figure 3(a) we plot $P^{(0)}$ versus $T_e$ and $d$. We observe the formation of narrow crests, separated by broad ‘valleys’, as in [3, 4]. The crests are formed when the Fermi energy touches the bottom of a 2D conduction band. In such a situation, $z_a - z_1$ is of the order of 1 (see the definitions after equation (22)), and the approximation (23) does not hold. In the valley regions, the approximation (23) holds and because of this $P^{(1)}_s$ and $P^{(1)}_a$ depend only on $T_{ph}$.

To confirm this, we calculate $P^{(1)}_s(T_e, T_{ph}), P^{(1)}_a(T_e, T_{ph})$, and $P^{(1)}(T_e, T_{ph}) = P^{(1)}_s(T_e, T_{ph}) + P^{(1)}_a(T_e, T_{ph})$ numerically. Indeed, in the valleys and in the temperature range investigated by us, both $P^{(1)}_s$ and $P^{(1)}_a$ are independent of $T_e$, whereas their dependence on $T_{ph}$ is the same as that of $P^{(0)}_s$ and $P^{(0)}_a$ on $T_e$, within the numerical accuracy. For this reason we do not also plot $P^{(1)}_s$, $P^{(1)}_a$ or $P^{(1)}(T_e, T_{ph})$ versus $T_{ph}$ and $d$. Numerically, these plots would be identical with the plots in figure 3.

In figure 3(b) we plot separately the contributions of the symmetric and antisymmetric phonon modes, $P^{(0)}_s$ and $P^{(0)}_a$, respectively, for $d = 10$ nm. For this thickness of the metallic layer, both $P^{(0)}_s$ and $P^{(0)}_a$ are well outside the crest regions and have typical (valley) behaviors. We observe that the antisymmetric modes have the dominant contribution at low temperatures (as seen also in [3, 4]); then, around $T_C$, both $P^{(0)}_s$ and $P^{(0)}_a$ exhibit crossovers. At higher temperatures (above $T_C$), $P^{(0)}_s$ and $P^{(0)}_a$ become gradually indistinguishable in figure 3(b).

In figure 4 we plot the exponent of the temperature dependence of $P^{(0)}_s$ (for $d = 10$ nm), denoted simply as $x = x_s = x_T$ (black dashed line). In the same figure we plot also the exponents of the temperature dependence of both $P^{(0)}_s$ and $P^{(0)}_a$:

$$x_s \equiv \frac{\partial \ln(P^{(0)}_s)}{\partial \ln(T_e)} = \frac{\partial \ln(P^{(0)}_s)}{\partial \ln(T_{ph})}$$
$$x_a \equiv \frac{\partial \ln(P^{(0)}_a)}{\partial \ln(T_e)} = \frac{\partial \ln(P^{(0)}_a)}{\partial \ln(T_{ph})}$$

(25)

While the main plot spans a temperature range of $[0, 20]$ K, to show the general behavior of $x$, $x_s$, and $x_a$ at both low and high temperatures, in the inset we show the details in a smaller temperature range (the small bump at 5 K is a numerical artifact due to the change of steps in the calculation of the numerical derivative). In the low-$T$ limit, we recover the results of [3, 4], namely $x = x_s = 3.5$ and $x_a = 4$. As the temperature increases, $x$ and $x_a$ increase, whereas $x_s$ first decreases and then increases. The bump of $x_s$ observed at $T_e$ around 400 mK can be correlated with the one we observe in $C_V$ (figure 2) and therefore it is caused by an increase in the DOS of the symmetric phonon gas. Starting from about 1.2 K and going up in temperature, the results for $x$, $x_s$, and $x_a$ become indistinguishable. This is similar to what happens for $C_{Vs}$ and $C_{Va}$ and therefore it is due to a similar dependence on energy of the DOS of the $s$ and $a$ modes—more exactly, the thermal energy gradually approaches the region where $\gamma_s \propto e^2$ (we shall come back to the dependence on energy of the DOS in section 4). At $T_e$ about 1.5 K, the exponents reach a ‘plateau’. After this plateau, from about 2.5 K up, the exponents decrease again, reach a minimum at $T_e \approx 12$ K, and then increase, approaching 4. Calculating the heat exchange at higher temperatures becomes increasingly difficult and time consuming, due to the poor convergence of the integrals.

4. Discussion

We observe that in the low-temperature limit, $x$ and $x_a$ converge to 3.5, confirming the low-temperature approximations of [3, 4], but not the general conjecture (1b) [6]. Nevertheless, as seen also in figure 2, the 2D limit for anti-symmetric phonons is actually reached at temperatures much lower than $T_C$ (in our case, $x_{ca}$ approaches 1 at temperatures of the order of 10 mK, whereas $T_C \approx 237$ mK). This is because the quadratic approximation of the dispersion relation of antisymmetric modes is valid only at energies much lower than $k_BT_C$.

At higher temperatures, one would expect that the 3D limit is reached and $x = x_a = x_T = 5$. As we can see in figure 2, the phonon gas practically reaches (or is very close to) the 3D limit at temperatures of the order of $10 T_C$. However, in figure 4 we observe that the 3D limit for $x$, $x_a$, and $x_T$ is not reached even at 20 K, which is already one order of magnitude above the temperature where the phonon gas is
practically 3D, so this must have other causes than the dimensionality of the phonon gas distribution. In order to qualitatively understand this effect, we investigate equations (19)–(24) in more detail.

In equation (20) we intentionally keep the terms \( \sum_{x} \int_{\phi_{x}} d\phi \) separate under the integrals. At temperatures where the phonon gas has a 3D distribution (say, above 10\( T \)), 

\[
\int_{\phi_{x}} d\phi \rightarrow \int_{\phi_{x}} d\phi \int_{\phi_{x}} d\phi \int_{\phi_{x}} d\phi
\]

may be approximated by a double integral which represents the integral over the phonon states and may be further transformed into an integral over the phonon energies, of the form

\[
\int_{\phi_{x}} d\phi \int_{\phi_{x}} d\phi \int_{\phi_{x}} d\phi
\]

as in equation (2). B u t s i n c e w e i n t e n t i o n a l l y k e e p t h e t e r m s \( \sum_{x} \int_{\phi_{x}} d\phi \) separate under the integrals. At temperatures where the phonon gas has a 3D distribution (say, above 10\( T \)), 

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\[
\int_{\phi_{x}} d\phi \rightarrow \int_{\phi_{x}} d\phi \int_{\phi_{x}} d\phi \int_{\phi_{x}} d\phi
\]
the phonon energy \( \epsilon_{\text{ph}} \equiv h\omega_{x_j} \) (see the appendix for the method of calculation). These averages are plotted in figure 6 and we see that they also have complicated structures, although a rough quasi-linear dependence on energy may be inferred, which could be correlated to the exponent \( x \approx 4 \) of the temperature dependence of \( P_x^{(0)} \) and \( P_x^{(1)} \) in the higher-temperature range plotted in figure 4.

The same discussion may be repeated almost identically for \( P_s^{(1)} \) and \( P_a^{(1)} \), with the only difference being that \( T_c \) is replaced by \( T_{\text{ph}} \).

Eventually, the 3D limit, with \( x = x_i = x_a = 5 \), is achieved when \( k_B T_c \) or \( k_B T_{\text{ph}} \) is comparable to or bigger than the energy difference between conduction bands. However, this limit is hard to achieve in our calculations due to numerical inaccuracies and the slow convergence of the integrals and the sums.

5. Conclusions

We have studied the effect of the dimensionality of the phonon gas distribution on the heat exchange between electrons and phonons in a layered structure consisting of a metallic layer (Cu) of a thickness around 10 nm deposited on an insulating free-standing membrane. The total thickness of the system (metal plus supporting membrane) is \( L = 100 \) nm. From a mechanical point of view (elastic properties and mass density) the system is considered homogeneous—the elastic waves propagate through a homogeneous material, making no difference between the metal and the dielectric membrane. The longitudinal sound velocity, the transversal sound velocity, and the density of the system are considered to be those of silicon nitride. With these parameters, the crossover temperature around which the phonon gas distribution changes from 2D (at lower temperatures) to 3D (at higher temperatures) is \( T_C \approx 257 \) mK [11].

To emphasize the change of dimensionality of the phonon gas distribution in a quantitative way, we plot in figure 2 the exponent of the temperature dependence of the heat capacity \( C_V \) (denoted as \( x_C \)) versus temperature. The change in dimensionality is reflected in the change of this exponent and we observe that this happens in a temperature interval around \( T_C \), as qualitatively explained in the introduction. Furthermore, \( x_C \) provides us with information about the overall DOS of the phonons in the energy interval which is thermodynamically relevant for a specific temperature. This information is used in section 4, to better understand the behavior of the heat exchange.

The heat power \( P \) exchanged between electrons and phonons is expected to be described by the ansatz (1a), with the exponent \( x \) satisfying the conjecture (1b). It was already noted that the conjecture does not apply to our type of layered system and the exponent corresponding to the 2D phonon gas distribution (that is, in the low-temperature limit) is \( x = 3.5 \), not 4 [3, 4]. We recover here the results of [3, 4] in the low-temperature limit (we obtain them numerically, without employing asymptotic analytic approximations) and we expect that, as the temperature increases and the phonon gas distribution changes from 2D to 3D, \( x \) should change continuously from 3.5 to the value corresponding to 3D systems, which is 5 [5]. Instead of this, we observe that \( x \) does not have a monotonic behavior and disobey the conjecture (1b) as well. The exponent starts from \( x = 3.5 \) at low temperatures, increases through the crossover region of temperatures around \( T_C \) and reaches a ‘plateau region’ in which \( x \) remains in a small interval, located between 4.5 and 5. The plateau region lies in a temperature range roughly between 0.5 and 3.5 K. After the plateau, the exponent decreases and around 7.5 K becomes smaller than 4 and remains so up to 20 K, which is the highest temperature used in our computations. This behavior of \( x \) is discussed in section 4 and it becomes clear that the dimensionalities of the phonon gas and of the electron gas are not the only ones that are important in the temperature dependence of the heat exchange. In our system, the energy dependence of the interaction term has a complicated structure and this is reflected in the temperature dependence of \( x \).

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Appendix. Calculation of the averages \( \langle I_{n,i}^{(0)}(x_j)/2x_j \rangle \)

To calculate the average of \( I_{n,i}^{(0)}(x_j)/2x_j \) as a function of energy, we divide the phonons’ energy axis into small intervals \( \delta\epsilon_{\text{ph}} = k_B \times 10 \) mK and the \( x_j \) axis into \( \delta x_j = 2 \times 10^{-4} \) intervals, forming the series \( \{\epsilon_{\text{ph},0}, \epsilon_{\text{ph},1}, ...\} \) and \( \{x_i,0, x_i,1, ...\} \), where \( \epsilon_{\text{ph},0} = 0 \) and \( x_i,0 = 0 \). The averages are calculated separately for each polarization \( \alpha = s, a \) in each of the intervals \( \{\epsilon_{\text{ph},j}, \epsilon_{\text{ph},j+1}\} \), using all the values \( I_{n,i}^{(0)}(x_j)/2x_j \) which correspond to this interval—that is, for all \( i \) and \( j \), such that \( h\omega_{x_j,x_i} \in \{\epsilon_{\text{ph},j}, \epsilon_{\text{ph},j+1}\} \).

In figure A1 we plot the counts that we obtain in each of the intervals \( \{\epsilon_{\text{ph},j}, \epsilon_{\text{ph},j+1}\} \), as a function of \( \epsilon_{\text{ph},j} \).
Figure A1. The counts in each of the energy intervals $\delta \epsilon_{ph} = k_B \times 10 \text{ mK}$, as a function of energy, for symmetric and antisymmetric phonons.

ORCID iDs

Dragos-Victor Anghel © https://orcid.org/0000-0003-4809-0482

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